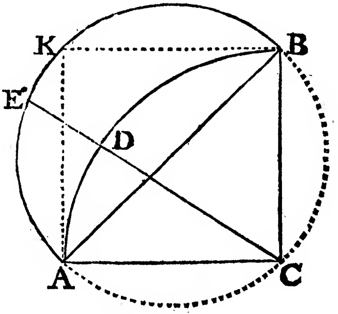

A Letter of Dr Wallis *to Dr* Sloan, *concerning the* Quadrature *of the* Parts *of the* Lunula *of* Hippocrates Chius, *performed by Mr* John Perks; *with the further* Improvements *of the ſame, by Dr* David Gregory, *and Mr* John Caſwell.

SIR,

THE *Squaring* a certain *Lunula* by *Hippocrates Chius* long ſince, hath been known (as to the whole *Lunula*) for many Ages. But (as to the *Parts* of it, and the *Appurtenances* thereunto,) *New* Diſcoveries have been lately made, which (I think) had not been conſider'd by any before this preſent Age.

I received (in *November* 1699.) from Mr. *John Perks* (Maſter of an Hoſpital at *Old-Swynford* in *Worceſter-ſhire*, founded by Mr, *Thomas Foley*) a brief account of his *Squariug* the *Portions* of *Hippocrates*'s *Lunula*; with which (I preſume) you will not be diſpleaſed.

For the better underſtanding of which; I ſhall premiſe as known, (becauſe long ſince demonſtrated,) That, If on AB (the



Chord of ADB, the Quadrantal Arc of a Greater Circle, who's Center is C,) be deſcribed, as on a Diameter, a Semi-circle ABE;

This Semi-circle, will be Equal to that Quadrant. (Becauſe the Squares of their Diameters, are as 2 to 1 ; And, in ſuch proportion are their reſpective Circles ; and therefore a Quarter of the one, equal to Half the other.)

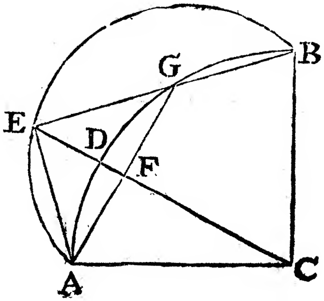
And, conſequently; If, from each of theſe, we ſubtract the common Segment ABD : the *Remaining Lunula* ADBE (on the one ſide) will be Equal to the *Remaining Triangle* (on the other ſide) ABC. (Or, to ABK, ſuppoſing AB biſected in K ; that is, to half the Square CK, inſcribed in the Leſſer Circle.) Which is commonly called, *The Squaring of Hippocrates's Lunula* ; That is, the Finding a *Rectilinear Figure* (which may be eaſyly reduced to a Square) equal to that *Lunula*.

This being premiſed ; The Point in hand, is, the *Squaring* a given *Portion* of ſuch *Lunula* : ſuppoſe ADE, cutt-off by a Streight Line CDE, drawn from the Center C. Which Mr *Perks* (not knowing that the like had been before attempted by any other) doth perform after this manner ; *viz.*

Drawing the Streight Lines EA, and EB (cutting the Arc EB in G,) and, on AG, a perpendicular EF, (which will therefore paſs to the Center C, becauſe Biſecting AG at Right-angles ;) *The Right-lined Triangle* AFE, *is equal to* ADE, *the propoſed Portion of the Lunula.*

His Demonſtration is to this purpoſe : *viz.*

ADB being a Quadrantal Arc ; the Angle AGB will be *Three Halves* of a Right Angle ; (and its Conjunct Angle EGA, *Half* a Right Angle.) And that Angle (being External to the Triangle AGE,) is Equal to the Two Oppoſite Internals GEA + EAG. Whereof GEA (becauſe an Angle in the Semicircle AEB), is a Right Angle ; and therefore EAG is *Half* a Right Angle, (as are alſo FEG, and FEA.) And the Three Triangles AFE, GFE, and



and GEA, each of them *Half a Square*. And AG to AE, as $\sqrt{2}$ to 1 (proportional to the Reſpective *Radii* of the Two Circles.) And the Like Segments ADG, AE, in their Reſpective Circles (as the Squares of their Reſpective *Radii*) as 2 to 1. And therefore the Semi-ſegment AFD, equal to the Segment AE. And conſequently (one taking from the Triangle as much as the other addes to it) the *Portion of the Lunula* ADE, equal to the *Triangle* AFE. Which was to be Demonſtrated.

(I take the liberty (both in this and the things that follow) to vary ſomewhat from the Authors Words, (but to the ſame ſenſe, and without any diſadvantage to Them,) ſo as to Deſign the ſame Reſpective Points (in all the Figures) by the ſame Letters. Which makes it ſomewhat Shorter (without Repeating the ſame Conſtruction anew for every Figure;) and prevents the Confuſion which might ariſe to the Fanſy, if the ſame Reſpective Points, in ſeveral Figures, were deſigned by different Letters; and the ſame Letters, in the different Figures, deſign different Points.)

If the Point E chance to be in K (the middle of the Arc AEB) there will be no Interſection at G (the Points G, B being then coincident, but without any diſturbance to the Demonſtration:) If it happen beyond it, toward B; then G will be on the other ſide; and what is here ſayd of EGB, muſt be accommodated to EGA: which things are ſo obvious, as not to need any long diſcourſe.

The whole proceeds upon the ſame general notion with that of ſquaring the whole *Lunula* (and ſome other Curve-lined Figures;) that, if as much be added to the one ſide, as is taken from the other, the Equality remains.

And the ſtreſs of the Demonſtration, is, to prove the ſegments ADG and AE, to be *Like Segments*; and therefore Proportional to their Reſpective Circles; the Whole of one, equal to Half the other.

The Ground of the whole Proceſs is plainly this, The Angle ACE, being an Angle at the Center of the Greater Circle, but at the Circumference of the Leſſer, the line CDE (as it paſſeth from CA to CB) doth, in the ſame proportion, divide the Quadrantal Arc ADB, and the Semicircular AEB: whence all the reſt doth naturally follow.

And this is Applicable to other *Lunula*'s (beſide that of *Hippocrates*) if (by altering the Angle at F, or otherwiſe,) we take-in ſuch a Portion of the common Segment ABD on the one ſide (inſtead of AE cut-off on the other ſide) as the Proportion of the two Circles requires.

I ſhewed this Quadrature of Mr. *Perks* to Dr. *David Gregory* (our learned Profeſſor of Aſtronomy at *Oxford*,) who gives his Opinion about it (with his Improvement of it) in a Letter of his to me ; which I ſhall give you in his own words,

“ Reverend Sir, The *Quadrature* of the *Parts* of the *Lunula* of *Hippocrates Chius*, by Mr. *Perks* (which you ſhewed me) is very Elegant.

“ I remember, the like was done, ſome years ſince, by Monſieur *Tchirnhauſe* ; who aſſigns, as equal to the ſame Portion, not the ſame Triangle with that of Mr. *Perks*, but another Equivalent thereunto, (as I ſhall ſhew by and by.) We have his *Theorem*, in the *Acta Lipſiæ*, for the Month of *September*, 1687. But, without any *Demonſtration*.

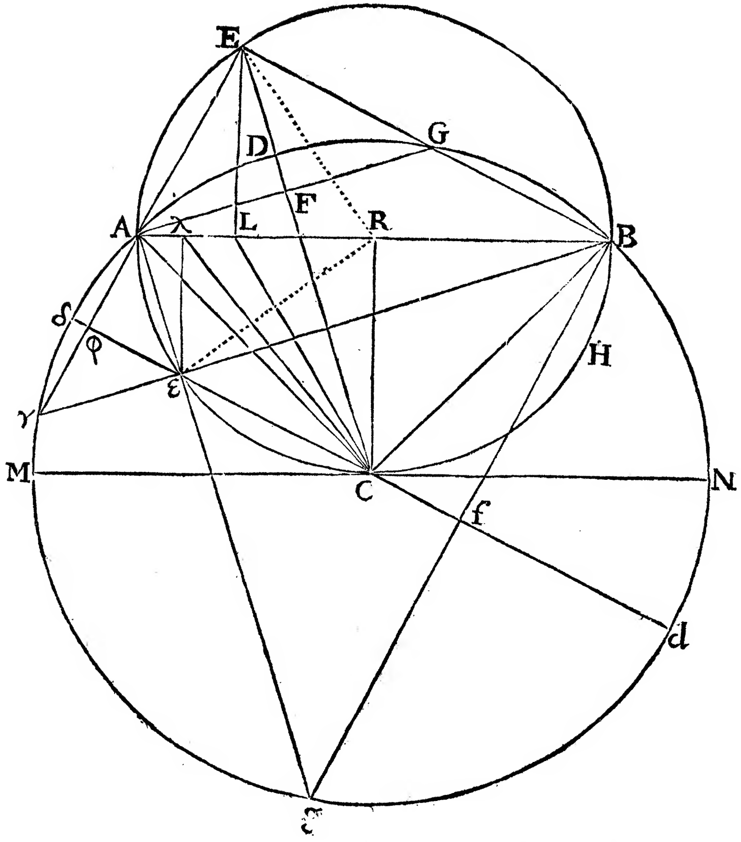
“ But, both the One and the Other, ſeem not to have conſidered this affair in its full extent.

“ For, if you compleat the Two Circles, whoſe Arcs contain the *Lunula* of *Hippocrates*; the ſame is true, as well of the Points in the other Semi-circle ACB, as of thoſe in the Semi-circle AEB; and, for the ſame Reaſons. As appears in the Scheme annexed, wherein I have mark'd the Points in the Semi-circle ACB, (correſpondent to thoſe of Mr. *Perks* in AEB,) with the correſpondent ſmall Letters of the Roman and Greek Alphabets.

“ If Mr. *Perks* had made his conſtruction univerſal; by making both EA and EB, meet with the Greater Circle, (which he might have done by protracting theſe Lines and the Greater Circle 'till they meet ;) he might have found that the Portions of the Spaces A ε CM, BHCN, (ſuppoſing MCN parallel to AB) are Quadrable as well as thoſe of *Hippocrates*'s *Lunula* : And that E A γ being a ſtreight Line, the Portion AED of *Hippocrates*'s *Lunula*, is to A ε δ (the Correſpondent of A ε C M) in the Duplicate Proportion of C ε to A ε. For E R ε (at R the Center of the Leſſer Circle) is, in this caſe, a Right Angle.

“ Moreover ; If you take any Point ε in the Semi-circle ACB, and proceed according to Mr. *Perk*'s conſtruction Univerſalized as above-ſaid ; you will find, on the one ſide, the *Trilineum* A ε δ (contained by the Arcs A ε, A δ, and the ſtreight line ε δ) equal to the Rectilineal Triangle A ε φ. And, on the other ſide, the *Trilineum* contained by the Arc B ε (the Complement of ε A to the Semi-circumference,) and the Arc B d (the Complement of A δ to the Fourth part of the Circumference,) and the ſtreight line ε d, (that is, the *Trilineum* BHCd diminiſhed by the Se-

“ gment



“ gment Cε;) to be equal to the Rectilineal Triangle Bεf. And,
“ that thoſe two ſpaces Aεδ, and the *Difference* of BHCd from
“ the Segment Cε (parts of the *Lunula* ACBgγA) taken to-
“ gether, are equal to the Triangle ACB; as well as the two
“ Spaces AED and BED, parts of the *Lunula* of *Hippocrates*.

“ So that, upon the whole, it appears, that the Two Circles
“ (containing the *Lunula* of *Hippocrates*) being completed; this
“ *Lunula* AEBGA, and the other ACBgγA, make up one Syſtem,
“ and are *Conjugate* Figures.

“ For, (drawing a ſtreight line CDE, or Cεδ, or Cεd, at pleaſure
“ through C the Center of the Greater Circle, and cutting thoſe
“ two Circles,) the Space contained within two Arcs of theſe two
“ Circles and part of the ſaid ſtreight line, (as AED, or Aεδ, or
“ BHεd,)

“BHεd,) is equal to the Rectilineal Triangle AEF, or Aεφ, or Bεf, reſpectively.

“ And it ſo happens, that, if this line going out from C, be on the ſame ſide of the Diameter MN with the *Lunula* of *Hippocrates*; the foreſaid Space (which receives a perfect Quadrature) is ſolitary; (ſuch as are the Parts of *Hippocrates*'s *Lunula*; and of the two Spaces AεCM, BHCN; which therefore are Parts of the *Lunula* more nearly relating to one another.)

“ But if that Line going out from C, be on the other ſide of MN; then the Space which is equal to the Rectilineal Triangle, is, the *Difference* of two Mixtilineal Figures, (the one a Trilineum, the other a Segment of the Leſſer Circle,) as is abovefaid; neither of which can be ſquared ſeverally.

“ All theſe particulars are plain from Mr. *Perks*'s Demonſtration; which, with a little variation (ſuch as is uſual in the different *Caſes* of the ſame *Theoreme*) is applicable to all of them: though perhaps he was not aware of it.

“ In the Dimenſion of the Parts of *Hippocrates*'s *Lunula*, it might perhaps be expected, that the Triangle aſſigned equal to a Portion of the *Lunula*, ſhould be Part of the Triangle to which that whole *Lunula* is wont to be aſſigned equal; (that is, that the Triangle aſſigned equal to the Portion ADE, ſhould be the reſpective part of ACB which is equal to the whole *Lunula*;) which in that of Mr. *Perks* is not.

“ But, in that of Mr. *Tſchirnhauſe* (above-mentioned) it is ſo, which is to this purpoſe.

“ If from any Point E, in the circumference of the Leſſer Circle, we let fall on AB, a Perpendicular cutting it in L, and draw the line CL; the Triangle CAL, is equal to the Portion of the *Lunula* AED. (And, conſequently, the Triangle CBL, equal to the Portion BED.)

“ Which (becauſe Mr *Tſchirnhauſe* hath not at all done it) I ſhall briefly Demonſtrate, ſo as the Demonſtration may reach the *Portions* of the *Conjugate* Space ACBgγA.

“ For the Triangles ACB, AEF, are like Triangles, each being the half of a Square: And therefore, by 19 el. 6, the Triangle ACB is to the Triangle AEF in the duplicate proportion of BA to AE, that is, by 8, el. 6, as BA is to AL. But, by 1. el. 6, the Triangle ACB is to the Triangle ACL as BA is to AL. Therefore, by 9. el. 5, the Triangles ACL and AEF are equal. But the Triangle AEF is (by Mr *Perks*) proved equal to the Portion

" tion AED. And therefore the ſaid Portion AED is alſo equal
" to the Triangle ACL.

"I am, Sir, Your &c. D. *Gregory*.

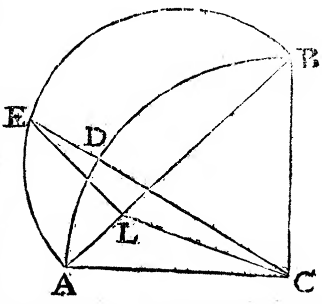
Mr *Caſwell* had a ſight of this Quadrature of Mr *Perks* (before Dr *Gregorie* or I had ſeen it ;) And had given a *Specimen* of its being capable of further Improvement. But, without having Leiſure, or giving himſelf the Trouble, of purſuing it through all its Appendages. I would (with his leave) have here inſerted that *Specimen*: But he choſe rather to decline it ; ſaying, He thought it needleſs, becauſe Dr *Gregorie* had, ſince, done the like more fully.

The Reſult of it, is to this purpoſe ; On the Center B, he draws by A, a Third Circle ; which forms another *Lunula* , than that of *Hippocrates* : And he doth (very dextrouſly) Square the *Portions* of this *Lunula*. And doth thereby let us in, to a New Syſtem, which may be purſued in like manner as Dr *Gregorie* hath done that of *Hippocrates*.

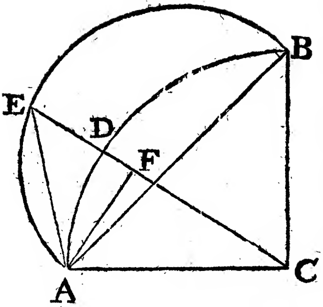
After theſe learned Diſquiſitions, on ſo trite a Subject ; it will not be needful for me to ſay much. I ſhall but briefly Compare the Two Quadratures of Mr *Tſchirnhauſe* and Mr *Perks*, (wherein they Agree or Differ with each other.) And then ſhew, How, by either of them, to Divide the *Lunula* in any Given Proportion.

Monſieur *Tſchirnhauſe* ; Letting fall, from E (on AB) a Perpendicular EL, determines the Triangle ALC equal to the Portion ADE.

Which being admitted ; We may thus Divide the *Lunula* in any Given Proportion. If we divide AB, at L, in ſuch Given Proportion ; CL will, in the ſame proportion (becauſe of the Common Altitude) divide the Triangle ACB (which is equal to the Whole *Lunula*.) And LE (erected at Right Angles on ALB) will determine the Point E ; from whence if we draw, to C, the Streight line EC, this will, at DE, divide the *Lunula* in the ſame Proportion.



Mr *Perks* ; On EDC , drawing the Perpendicular AF , determines the Semi-quadrate AFE , equal to the propoſed Portion



tion ADE. Which Semi-quadrate, is a *Like Figure*, and *a like ſituate* to AE, as is ACB to AB.

And therefore (becauſe like Figures are in the Duplicate Proportion of their reſpective Sides) If we ſo inſcribe AE, as that the Square of AE be to the Square of AB, in ſuch Given Proportion, the *Lunula* will at DE, be ſo divided as is required.

And this will hold (if duly applied, according as the different Caſes may require) though E be taken (in the Continuation of the Semi-circle) beyond B. For (ſtill) Like Figures, will be in Duplicate Proportion of their Reſpective Sides; and $CE = CD \pm DE$. And the ſame is yet improveable much further.

I forbear to Apply this to the ſeveral Parts of the whole Syſteme, conſidered by Dr *Gregorie*, (Or to that of Mr *Caſwell*,) that I be not too Teadious.

Much leſs ſhall I give my ſelf the trouble to conſider the Solids to be made by the Converſion of it, or of its parts, about a given Axis, (as MN, or AB, or AC, or BC, &c.) with their Surfaces and Centers of Gravity; as I have done elſewhere for the *Cycloid*: But ſuch as are at Leiſure (and think it worth the while,) may do it by ſuch like Methodes as I have made uſe of for the Cycloide,

I am *SIR*,
Yours to ſerve you,
JOHN WALLIS.

Poſt-ſcript.

In the *Tranſactions* for the Month of *Auguſt* laſt paſt; *Numb*. 255. A Letter of mine, is very faultyly Printed. I deſire that the *Errata* may be thus Corrected.

Pag. 280. *l*. 24. ut ait. *p*. 281. *l*. 15. differentias infiniteſimas. *p*. 282. *l*. 12. (ut antea) rerum Novitas. *l*. 14. Meſſis. *l*. 15. Et quidem. *l*. 16. Atque hinc. *l*. 17. natura. *l*. 22. Academia. *l*. 25. reapſe. *l*. 33. miſi. *p*. 283. *l*. 5. deſperatum. *l*. 11. Sueci. *l*. 17. itinere. *l*. 25. adornat. *l*. 33. Cœno. *p*. 284. *l*. 1. ſita. *l*. 13. redeundo) ſenſim. *l*. 17. motibus. *l*. 19. penitius. *l*. 22. materiæ. *l*. 23. perpendiculum erectos) ad. *l*. 24. longo tractu. *l*. 25. præ ſe. *l*. 29. Multaque. *l*. 30. annos. *l*. 31. deprompta) mihi videntur huc. *l*. 32. aliò. *l*. 34. cœnoſum, turbidum. *l*. 35. Iſthmo. *l*. *ult*. *The Words* P. S. Aug. 29. 1699. *ſhould have ſtood at lin*. 20. /

Numb. 257. *p*. 346. *l*. 11. the *Solar Tropical* year. *p*. 349. *l*. 2. ſuggeſted by. *p*. 351. *l*. 34. ſtands thus.